Elastic Scattering Based on Integral Equation Theory for Potentials Including the Coulomb Potential

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ABSTRACT

The upper bounds for the regular Coulomb wave function and the Green function are involved with the integral equation. Using the uniform convergence of the integral equation for the wave function, it is found that the wave function is an analytic function of k except at $k = 0$. In this respect it will be shown that S-matrix element is an analytic function of k except at $k = 0$ and at it poles.

KEYWORDS: Analytic, Elastic Scattering, Poles, S-Matrix, Wave Function

INTRODUCTION

This research is incorporated the Coulomb potential to the elastic scattering theory through a Volterra type integral equation (Levinson, 1949) which directly follows from the corresponding Schrödinger equation. It will be shown that this integral equation for the wave function is uniformly convergent with respect to the radial distance r and formulated the elastic S-matrix element using the asymptotic form of the wave function for the full fledged potential with the Coulomb potential. Using the corresponding integral equations for the expressions involved in the elastic S-matrix element it will be shown that S-matrix element is an analytic function of k except at $k = 0$ and at it poles.

METHODOLOGY

Wave Function and its Analytic Properties

The Schrödinger equation relevant to this research discussion takes the form

$$u_l(r,k) = \frac{F_l(kx)}{k^{l+1}} - \frac{1}{k}\int_0^r g(r,\xi,k)V(\xi)u_l(\xi,k)d\xi, \quad (1)$$

where

$$g(r,\xi,k) = F_l(kr)G_l(k\xi) - F_l(k\xi)G_l(kr) \quad (2)$$

and the well known factor $\frac{2\mu}{\hbar^2}$ is included in the potential V itself where μ is the reduced mass.

$F_l(\rho) \approx C\rho^{l+1}\phi_l(\rho)$, where C is a constant, and $F_l(\rho)$ is the regular solution of the Schrödinger equation corresponding to the Coulomb potential.

$$F_l''(\rho) = -\left[1 - \frac{2\eta}{\rho} - \frac{l(l+1)}{\rho^2}\right]F_l(\rho) \quad (3)$$

where $\rho = kr$ and $\eta = \frac{4\pi z_1 z_2 e^2}{\hbar^2 k}$, z_1, z_2 being the proton numbers of the two nuclei scattering on each other.

It should be noted that, this research is interested in the scattering of point like charged particles. Studying of scattering of bound states having a volume can also be treated along the same lines.

This research is used of the following result on differentiation under the integral sign (Gradshteyn & Rtzhik, 1980).

$$\frac{d}{d\alpha}\int_{\xi(\alpha)}^{\varphi(\alpha)} f(x,\alpha)dx = \frac{d\varphi}{d\alpha}f(\varphi(\alpha),\alpha) - \frac{d\xi}{d\alpha}f(\xi(\alpha),\alpha) + \int_{\xi(\alpha)}^{\varphi(\alpha)}\frac{\partial f}{\partial\alpha}dx \quad (4)$$

Let us first show that the wave function $u_l(k,r)$ satisfies the Volterra type integral equation

$$u_l(r,k) = \frac{F_l(kr)}{k^{l+1}} - \frac{1}{k}\int_0^r g_l(r,\xi,k)V(\xi)u_l(\xi,k)d\xi$$

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$$\frac{d}{dr}g(r,\xi,k) = \frac{d}{dr}F_l(kr)G_l(k\xi) - F_l(k\xi)\frac{d}{dr}G_l(kr),$$

$$F'_l(kr)G_l(kr) - F_l(kr)G'_l(kr) = 1 \quad (5)$$

$$\frac{d^2}{dr^2}g(r,\xi,k) = k^2[F''_l(kr)G_l(k\xi) - F_l(k\xi)G''_l(kr)] = k^2 D(\rho), \quad (6)$$

where

$$D(\rho) = -\left[1 - \frac{2\eta}{\rho} - \frac{l(l+1)}{\rho^2}\right][F_l(kr)\, G_l(k\xi) - F_l(k\xi)G_l(kr)]$$

$$\frac{d^2}{d^2\alpha}\int_{\xi(\alpha)}^{\varphi(\alpha)} f(x,\alpha)dx = \frac{d\varphi}{d\alpha}f'(\varphi(\alpha),\alpha) + \int_{\xi(\alpha)}^{\emptyset(\alpha)} \frac{\partial}{\partial x}\frac{\partial f}{\partial x}dx \quad (7)$$

$$\frac{d^2}{dr^2}u_l(k,r) = -\left[k^2 - \frac{2\eta k}{r} - \frac{l(l+1)}{r^2} - V(r)\right]u_l(k,r) \quad (8)$$

Asymptotic form of $u_l(k,r)$ for the complex potential with the Coulomb potential

The series expansion for $F_l(\rho)$ and $G_l(\rho)$ can be written down as (Flugge, 1959),

$$F_l(\rho) = \frac{1}{2}|[\frac{e^{\frac{\pi i x}{2}}\Gamma\left(\frac{1}{2}+m+x\right)}{\Gamma(2m+1)}]|e^{-\pi i/2(m+\frac{1}{2})}M_{x,m}(z), \quad (9)$$

$$G_l(\rho) = , \frac{1}{2}|[\frac{e^{\frac{\pi i x}{2}}\Gamma\left(\frac{1}{2}+m+x\right)}{\Gamma(2m+1)}]|e^{-\pi i/2(m+\frac{3}{2})}\bar{M}_{x,m}, \quad (10)$$

where $z = 2i\rho$, $x = i\,\eta$.

$F_l(kr) = C_l(kr)^{l+1}\emptyset_l[kr]$, where

$C_l = \left|2^l e^{-\frac{\pi\eta}{2}}\frac{\Gamma(l+1-i\eta)}{\Gamma(2l+2)}\right|$ and

$$\emptyset_l(kr) = 1 + \frac{\eta}{l+1}(kr) + \frac{2\eta^2-(l+1)}{2(l+1)(2l+3)}(kr)^2 + \cdots \quad (11)$$

$$\emptyset_l(\rho) = \sum_{j=l+1} A_j\,\rho^{j-l-1}$$

$$A_j = \frac{2\eta A_{j-1} - A_{j-2}}{(j+l)(j-l-1)},$$

where $A_j = 0,\ j < l+1\ ,\ A_{l+1} = 1$.

If $|\eta|^2 < (l+1)^2 d^2$, i.e. $\frac{|\eta|}{l+1} < d$,

$|dk| = v\ ,\ \eta = \frac{4\pi e^2 z_1 z_2}{h^2 k}\ ,\ k \neq 0$,

it is implied that $|\emptyset_l(kr)| < 1 + \frac{d|(kr)|}{1} + \frac{d^2|(kr)|^2}{2!} + \frac{d^3|(kr)|^3}{3!} + \cdots$

$$|\emptyset_l(kr)| < e^{d|k|r} < C_l\frac{|kr|^{l+1}}{(1+|kr|)^{l+1}}\ e^{vr},$$

$r \geq 0$.

$|F_l(kr)| \leq C_l e^{vr}\frac{|kr|^{l+1}}{(1+|kr|)^{l+1}}$, for some constant C_l, and also for $r \geq \xi \geq 0$.

The asymptotic form of F_l takes the form

$F_l(kr) \sim \sin(kr - \eta\log 2kr - \frac{l\pi}{2} + \sigma_l(k))$,

where $\sigma_l(k) = \arg\Gamma(l+1+i\eta)$

is the constant Coulomb phase shift.

$G_l(kr)$ is defined so as to have an asymptotic phase differing by 90° from that of $F_l(kr)$ and normalized so that

$G_l(kr) \sim \cos(kr - \eta\log 2kr - \frac{l\pi}{2} + \sigma_l(k))$.

$$e^{i\sigma_l} = \left[\frac{\Gamma(l+1+i\eta)}{\Gamma(l+1-i\eta)}\right]^{\frac{1}{2}}, \quad (12)$$

$$F_l = \frac{1}{2i}(H_l - H_l^*), G_l = \frac{1}{2}(H_l + H_l^*), \quad (13)$$

where (Flugge, 1959)

$$H_l^* = i\exp\left[\frac{\pi i}{2}\left(m + \frac{1}{2} - x\right)\right]\left[\frac{\Gamma\left(m+\frac{1}{2}-x\right)}{\Gamma\left(m+\frac{1}{2}+x\right)}\right]^{\frac{1}{2}} W_{x,m}(z) \quad (14)$$

$$H = i\exp\left[-\frac{\pi i}{2}\left(m+\frac{1}{2}x\right)\right]\left[\frac{\Gamma\left(m+\frac{1}{2}+x\right)}{\Gamma\left(m+\frac{1}{2}-x\right)}\right]^{\frac{1}{2}} W_{-x,m}(-z) \quad (15)$$

when $m = l + \frac{1}{2}$, $x = i\eta$, $W_{x,m}(z) \sim e^{-\frac{z}{2}} z^x$.

When

$$g(r,\xi,k) = F_l(kr)G_l(k\xi) - F_l(k\xi)G_l(kr),$$

using above formulations, it gets,

$$g(r,\xi,k) = \frac{1}{2i}[H_l(kr)H_l^*(k\xi) - H_l^*(kr)H_l(k\xi)] \quad (16)$$

$$g(r,\xi,k) = \frac{1}{2i}\left[-W_{-x,m}(-kr)W_{x,m}(k\xi)e^{-\pi xi} + W_{x,m}(kr)W_{-x,m}(-k\xi)e^{-\pi xi}\right] \quad (17)$$

$$|g(r,\xi,k)| \le K\left|\frac{e^{-\pi xi}}{2i} e^{\frac{k}{2}(r-\xi)}\right|\left[\left|e^{-2k(r-\xi)}\right|\frac{|kr|^x}{|k\xi|^x} + \frac{|k\xi|^x}{|kr|^x}\right]\Big|$$

$$|kr|^{i\eta} = e^{i\eta\log|kr|} < e^{d\log|kr|},$$

$$|k\xi|^{i\eta} = e^{i\eta\log|k\xi|} = e^{d\log|k\xi|}$$

$$\left[\left|e^{-2k(r-\xi)}\right|\frac{|kr|^x}{|k\xi|^x} + \frac{|k\xi|^x}{|kr|^x}\right] < e^{2|k|(|r-\xi|+d\log\frac{r}{\xi}} + e^{-d\log\frac{r}{\xi}} < C_l e^{v(r-\xi)+d\log\frac{r}{\xi}}$$

it can be written as

$$|g(r,\xi,k)| \le K_1 e^{\frac{|k|}{2}(r-\xi)+\pi d+2|k|(r-\xi)+d\log\frac{r}{\xi}} < Ke^{v(r-\xi)+d\log\frac{r}{\xi}}$$

$$F_l(kr) = C_l(kr)^{l+1}\phi_l[kr]$$

$$G_l(kr) = D_l(kr)^{-l}\theta_l(kr),$$

$\theta_l(kr)$ is an ascending series starting with 1.

Finally it can write

$$|g(r,\xi,k)| \le Ke^{v(r-\xi)+d\log\frac{r}{\xi}}\frac{(1+|k\xi|)^l}{(1+|kr|)^{l+1}}\frac{|kr|^{l+1}}{|k\xi|^l}, r \ge \xi > 0 \quad (18)$$

In general, now it can write

$$|F_l(kr)| \le Ke^{vr}\frac{|kr|^{l+1}}{(1+|kr|)^{l+1}} \quad (19)$$

$$|g(r,\xi,k)| \le Ke^{v(r-\xi)+d\log\frac{r}{\xi}}\frac{(1+|k\xi|)^l}{(1+|kr|)^{l+1}}\frac{|kr|^{l+1}}{|k\xi|^l}, r \ge \xi > 0 \quad (20)$$

Analyticity of $u_l(k,r)$.

Analyticity of $u_l(k,r)$ depends on $F_l(k,r)$and $g(r,\xi,k)$. These functions are bounded for $k \ge 0$ by the following expressions,

$$|F_l(kr)| < Ke^{vr}\frac{|kr|^{l+1}}{(1+|kr|)^{l+1}}, \quad (21)$$

$$|g_l(r,\xi,k)| \le Ke^{v(r-\xi)+d\log\frac{r}{\xi}}\frac{(1+|k\xi|)^l}{|k\xi|^l}\frac{|kr|^{l+1}}{(1+|kr|)^{l+1}},$$

$$r \ge \xi > 0 \quad (22)$$

where K is some finite constant. Let us now consider a sequence of function $u_l(r,k)$ defined by

$$u_l^{(n)}(r,k) = \frac{F_l(kr)}{k^{l+1}} - \frac{1}{k}\int_0^r g_l(r,\xi,k)V(\xi)\, u_l^{(n-1)}(\xi,k)d\xi \quad (23)$$

$$u_l^{(0)}(r,k) = 0.$$

By iteration and using the inequalities (21), (22) we get

$$\left|u_l^{(n)}(r,k) - u_l^{(n-1)}(r,k)\right| \le \frac{1}{k}$$

$$\left|\int_{+0}^r Ke^{v(r-\xi)+d\log\frac{r}{\xi}}\frac{(1+|k\xi|)^l}{|k\xi|^l}\frac{|kr|^{l+1}}{(1+|kr|)^{l+1}}V(\xi)\left[u_l^{(n-2)}(\xi,k) - u_l^{(n-1)}(\xi,k)\right]d\xi\right|$$

Since,

$$|g_l(\xi,\xi,k)| \leq K e^{v(\xi-\xi)+d\log\frac{r}{\xi}} \frac{(1+|k\xi|)^l}{|k\xi|^l} \frac{|k\xi|^{l+1}}{(1+|k\xi|)^{l+1}}$$

$$= K\frac{|k\xi|}{(1+|k\xi|)} \qquad (24)$$

Then by some calculations it can get finally,

$$\left|u_l^{(n)}(r,k) - u_l^{(n-1)}(r,k)\right| = K^n \frac{|r|^{l+1}}{(1+|kr|)^{l+1}} e^{vr+d\log r}\left[\int_{+0}^{r} \frac{e^{-d\log\xi}\xi|V(\xi)|}{(1+|k\xi|)} d\xi\right]^{n-1},$$

$\xi > 0$.

Hence

$$\left|u_l^{(n)}(r,k) - u_l^{(n-1)}(r,k)\right| \leq K^n \frac{|r|^{l+1}}{(1+|kr|)^{l+1}} e^{vr+d\log r}[L(r)]^{n-1}, \qquad (25)$$

where $L(r) = \int_{+0}^{r} \frac{e^{-d\log\xi}\xi|V(\xi)|}{(1+|k\xi|)} d\xi\,, \xi > 0$.

Thus the sequence $u_l^{(n)}(x,k)$ approaches uniformly to the limit $u_l(x,k)$, provided $L(x)$ is finite. Each $u_l^{(n)}(r,k)$ is analytic, since it involves analytic function $F_l(kr)$ and $g_l(r,\xi,k)$ only. Hence the limit $u_l(r,k)$ is analytic in the upper half-plane.

Elastic S-matrix element

The wave function $u_l(k,r)$ is given by

$$u_l(r,k) = \frac{F_l(kr)}{k^{l+1}} - \frac{1}{k}\int_0^r g_l(r,\xi,k)V(\xi)u_l(\xi,k)d\xi$$

with the boundary condition

$$u_l(0,k) = 0.$$

As r tends to infinity,

$$u_l\,(k,r) = \frac{A_l(k)}{k^{l+1}}\sin(kr - \frac{l\pi}{2} + \theta_l(k)\,),$$

where $\theta_l(k)$ is the asymptotic phase which is a function of k. By the Schrödinger equation it follows that $u_l(-k,r) = u_l(k,r)$, $A_l(k)$ is an even function of k and $\theta_l(k)$ is an odd function of k. Let us define (Gradshteyn and Rtzhik, 1980).

$$F_l(k) = 1 + ik^l\int_0^\infty [F_l(k\xi) + iG_l(k\xi)]V(\xi)u_l(\xi,k)d\xi$$

$$= F_l^{(1)}(k) + iF_l^{(2)}(k), \qquad (26)$$

where

$$F_l^{(1)}(k) = 1 - k^l\int_0^\infty G_l(k\xi)V(\xi)u_l(\xi,k)d\xi \qquad (27)$$

$$F_l^{(2)}(k) = k^l\int_0^\infty F(k\xi)V(\xi)u_l(\xi,k)d\xi \qquad (28)$$

In case of the Coulomb potential, let

$$u_l(r,k) = \frac{F_l(kr)}{k^{l+1}} - \frac{1}{k}\int_0^r g_l(r,\xi,k)V(\xi)u_l(\xi,k)d\xi$$

$= \frac{1}{k^{l+1}}[F_l^{(1)}(k)F_l\,(\,kr\,) + F_l^2(k)G_l(kr)]$ and as $r \to \infty$,

$$= \frac{1}{k^{l+1}}[e^{-i(kr-\eta\ln 2kr-\frac{l\pi}{2}+\sigma_l)}(-\frac{1}{2i}F_l^{(1)}(k) + \frac{1}{2}F_l^{(2)}(k)) + e^{i(kr-\eta\ln 2kr-\frac{l\pi}{2}+\sigma_l)}(\frac{1}{2i}F_l^{(1)}(k) + \frac{1}{2}F_l^{(2)}(k)\,)]$$

Now it can be defined the elastic S-matrix element as

$$S_l(k) = \frac{F_l^{(1)}(k)+iF_l^{(2)}(k)}{F_l^{(1)}(k)-iF_l^{(2)}(k)}e^{2i\sigma_l} \qquad (29)$$

Since $F_l(kr) \to \sin\left(kr - 2\eta\ln(2kr) - \frac{l\pi}{2} + \sigma_l(k)\right)$ and $G_l(kr) \to \cos(kr - 2\eta\ln(2kr) - \frac{l\pi}{2} + \sigma_l(k)\,)$.

It is clear that both $F_l^{(1)}(k)$ and $F_l^{(2)}(k)$ are real when k is real and hence

$|S_l(k)| = 1$ follows for real k. Now, $F_l^{(1)}(k)$ and $F_l^{(2)}(k)$ are not analytic functions of k at $k = 0$.

It is, however, clear that the both $F_l^{(1)}(k)$ and $F_l^{(2)}(k)$ are analytic function of except at $k = 0$, and hence the elastic S-matrix element is an analytic function of k except at $k = 0$, which is an essential singularity, and at it poles.

CONCLUSION

The integral equation method is studied for complex potential with the Coulomb potential to derive analytic properties of the wave function and the elastic S-matrix element. First of all, the relation between the Volterra type integral equation and the Schrödinger equation is justified using simple relations of the Coulomb wave functions. In the next step, upper bounds for the regular Coulomb wave function and the Green function involved with the integral equation are derived. Using the uniform convergence of the integral equation for the wave function, it is found that the wave function is an analytic function of k in the upper half plane. In this respect it is found that the integral equation for the wave function converges uniformly and hence it is analytic function of k except at $k = 0$.

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