FOUNDATION TO LOGIC

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1.0 History

Elementary logic has been studied since ancient times, in part through the analysis of paradoxes, and as a part of rhetoric. Late in the nineteenth century Frege and others attempted to formalize mathematics and the laws of deduction. At the turn of the 20th century a debate arose regarding the legitimacy of non-constructive proofs; Hilbert suggested a program demonstrating the possibility of securing mathematics onto a formal foundation. (Whitehead and Russell (1910-1913) actually tried it.) The necessary groundwork in logic, outlined above, was laid during the 1920s and 1930s, which is when some of the most paradoxical results were obtained. The implicit dependence on set theory and the inability to determine a decidable set of first-order axioms for set theory have caused considerable consternation among mathematicians, particularly those confronted with difficulties associated with the axiom of choice. Notable among post-war developments is Robinson's application of model theory to develop nonstandard analysis and use it as a framework for ordinary calculus. Many undesirability results appeared in the 1960s and 1970s with applications in traditional branches of algebra. With the development of computer science during later decades, many topics in recursion theory and proof theory were developed from the perspective of the theory of algorithms.

2.0 Introduction

Mathematical Logic is the study of the processes used in mathematical deduction. The subject has origins in philosophy, and indeed it is only by nonmathematical argument that one can show the usual rules for inference and deduction (law of excluded middle; cut rule; etc.) are valid. It is also a legacy from philosophy that we can distinguish semantic reasoning ("what is true?") from syntactic reasoning ("what can be shown?"). The first leads to Model Theory, the second, to Proof Theory.

Students encounter elementary (sentential) logic early in their mathematical training. This includes techniques using truth tables, symbolic logic with only "and", "or", and "not" in the language, and various equivalences among methods of proof (e.g. proof by contradiction is a proof of the contrapositive). This material includes somewhat deeper results such as the existence of disjunctive normal forms for statements. Also fairly straightforward is elementary first-order logic, which adds quantifiers ("for all" and "there exists") to the language. The corresponding normal form is prenex normal form. In second-order logic, the quantifiers are allowed to apply to relations and functions -- to subsets as well as elements of a set. (For example, the well-ordering axiom of the integers is a second-order statement).

Mathematics uses certain words and phrases in a very special way. What is meant by a "**statement**" in mathematics? Grammatically, a statement must be a sentence. It must have a subject and predicate; furthermore, a statement must be either true or false.

2.1 Logic: Logic is study of reason. It is specially concern with whether reason is correct. It is focuses on the relationship among statement, but doesn't focus form the content of the particular statement.

2.2 Proposition: A statement or a sentence that is either true or false but not both. Propositions are the basic building block of material of logic. We normally use lower case letters to represent the propositions.

2.3 logical connectives: Suppose we have several statements, denote by p,q,r,..... We form new "compound" statement we connect the given statements by means of such words as "**or**", "**and**", "**if and only if**" , which are called logical connectives.

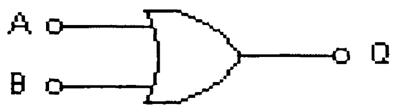
2.4 Disjunction: The logical connective or is denoted by the symbol "v"; "p or q" is written as "pvq" and is called the disjunction of p and q. The connective V is used in mathematics in an inclusive sense, so that pvq is true whenever at least one of p ,q is true; it is false only when both p and q are false. In applications this operation is doing by OR logic gate.

Truth Table

p	q	pvq
T	T	T
T	F	T
F	T	T
F	F	F

Table 1

OR logic gate



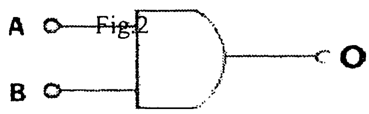
2.5 Conjunction: The logical connective and is denoted by the symbol "^" or "&"; "p and q" is written as "p^q" and is called the conjunction of p and q. The compound statement p^q is true when both p and q are true, otherwise it is false. In applications this operation is doing by AND logic gate.

Truth Table

p	q	p^q
T	T	T
T	F	F
F	T	F
F	F	F

Table 2

AND logic gate



2.6 Negation: The statement "not p" is called the negation of p or the denial of p and is denoted "¬p". The statement ¬p has a truth values opposite to that of p. In applications this operation is doing by NOT logic gate.

Truth Table

p	¬p
T	F
F	T

Table 3

NOT logic gate

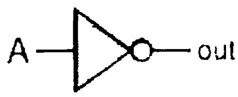


Fig. 3

2.7 Fundamental conjunction: We mean either (i) a literal or (ii) a conjunction of two or more literals no two of which involve the same statement letter. For instances, B', A&B are fundamental conjunctions, while (A')' , A&B&A are not fundamental conjunctions.

2.8 Included Conjunctions: One fundamental conjunction **A** is said to be included in another **B** if all literals of **A** are also literals of **B**. For example, A&B is included in A&B, B&B&C' is included in C'&B is included in A&B&C', while B is not included in A&B'.

3.0 Simplification Methods

These are the basic functions of logics. These logic gates are used in many electronics circuits. In real world problems we must work with very large amount of logic gates. But it is impossible. As mathematicians we can reduce the circuit using some algebraic operation. There are some simplification methods. Boolean algebra, Karnaugh Map and some algebraic methods are used to simplification'

3.1 Boolean Identity Laws

01	x+0=x	x.1=x	Identity law
02	x+1=1	x.0=0	Domination
03	x+x=x	x.x=x	Idemponent
04	x+x'=1	x.x'=0	Inverse
05	x+y=y+x	x.y=y.x	Commutative
06	x+(y+z)=(x+y)+z	x.(y.z)=(x.y).z	Associativity
07	x.(y+z)=x.y+x.z	x+y.z=(x+y).(x+z)	Distributive
08	(x')'=x		Complementary
09	(x+y)'=x'.y'	(x.y)'=x'+y'	DeMorgan's
10	x+x.y=x	x.(x+y)=x	Absorption

Table 4

3.2 Algebraic method

There is another method to simplify the statements. It is based on modulo 2 operations. In this method we have some properties.

Let p is a proposition.

$p + p = 2p = 0$
$1 + p = p'$
$p \cdot p = p^2 = p$

Table 5

Then we can derive the truth functions for basic logic gates using truth tables. And also here we use normal addition and multiplication.

AND logic gate

p	q	p&q
T	T	T
T	F	F
F	T	F
F	F	F

Table 6

Truth function of Logic AND gate = pq

OR logic gate

p	q	p v q
T	T	T
T	F	T
F	T	T
F	F	F

Table 07

Truth function = $pq + pq' + p'q$
$= pq + p(1+q) + (1+p)q$ [q' =1+q and p'=1+p]
$= pq + p + pq + q + pq$
$= 2pq + p + q + pq$
$= p + q + pq$ [2pq = 0]

Truth Function of logic OR gate = p + q + pq

For example, if p,q,r are propositions.

$pqr + p'qr + pq'r + p'q'r = pqr + (1+p)qr + (1+q)pr + (1+p)(1+q)r \quad [p' = 1 + p]$
$= pqr + qr + pqr + pr + pqr + (1+p+q+pq)r$
$= qr + pr + pqr + r + pr + qr + pqr \quad [2pqr = 2pr = 2qr = 0]$
$= r$

4.0 Representing Boolean Functions

4.1 Sum-of-products Expansions (SOP)
Also known as Disjunctive Normal Form (DNF)
4.2 Product-of-sums Expansions (POS)
Also known as Conjunctive Normal Form (CNF)
4.3 Functional Completeness: Minimal functionally complete sets of operators.

4.1 Sums-of-Products Expansions (DNF)

Any Boolean function can be represented as a sum of products of variables and their complements.
A **literal** is a Boolean variable or its complement, e.g. x,y,z,x',y',z' are all literals.
A **minterm** of Boolean variables x_1, K, x_n is a Boolean product of n literals y_1, K, y_n, where y_i is either the literal x_i or its complement x'_i A minterm is 1 for only one combination values of variables.
On the other words a statement from **A** is said to be in disjunctive normal form if either (i) A is a fundamental conjunction, or (ii) **A** is a disjunction of two or more fundamental conjunctions, of which none is included in another.
The following statement forms are in DNF

a) B
b) C' v C
c) A v (B'&C)
d) (A & B') v ((A'&B'&C)

4.2 Conjunctive Normal Form (CNF)

A **maxterm** is a sum of literals. Conjunctive Normal Form, CNF is a product-of-maxterms representation is sometimes called product-of-sums. The expression below is in CNF:

$F = (x+y+z).(x+y+z').(x'+y+z')$

Each Boolean function has a CNF: Take the dual of the sum-of-products representation.
Also a statement form **A** is in conjunctive normal form if either (i) **A** is a fundamental disjunction or (ii) **A** is a conjunction of two or more fundamental disjunctions of which included in another.
For Example:

a) (A v B v C')&(A v B)
b) A'

4.3 Functional Completeness

Since every Boolean function can be expressed in terms of •,+,¯, we say that the set of operators {•,+,¯} is **functionally complete**.
There are smaller sets of operators that are also functionally complete.

We can eliminate either • or + using DeMorgan's law. Thus {•,¯} is functionally complete and so is {+,¯}.
NAND | and NOR ↓ are also functionally complete, each by itself (as a singleton set).

E .g: x=x | x and x y =(x | y) | (x | y)

This article is mainly focused to give small idea about mathematical meaning of the Logic and the circuit simplification using Truth functions. Logic circuit simplification means the logic statement is reduced to the form of either Disjunctive normal form or Conjunctive normal form. The mathematicians job is do it. Then logic circuit designer can design the simple circuit. It is very useful because it reduced the time, cost, space and etc.